Positive plurisubharmonic currents: Generalized Lelong numbers and Tangent theorems

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Abstract:

Dinh–Sibony theory of tangent and density currents is a recent but powerful tool to study positive closed currents. Over twenty years ago, Alessandrini and Bassanelli initiated the theory of the Lelong number of a positive plurisubharmonic current in $\mathbb{C}^k$ along a linear subspace. Although the latter theory is intriguing, it has not yet been explored in-depth since then. Introducing the concept of the generalized Lelong numbers and studying these new numerical values, we extend both theories to a more general class of positive plurisubharmonic currents and in a more general context of ambient manifolds.

More specifically, in the first part of our article, we consider a positive plurisubharmonic current T of bidegree (p,p) on a complex manifold X of dimension k, and let $V \subset X$ be a Kähler submanifold of dimension l and B a relatively compact piecewise $\mathscr{C}^2$-smooth open subset of V. We impose a mild reasonable condition on T and B, namely, T is weakly approximable by $T_n^+ - T_n^-$ on a neighborhood U of $\overline{B}$ in X, where $(T_n^\pm)_{n=1}^\infty$ are some positive plurisubharmonic $\mathscr{C}^3$-smooth forms of bidegree (p,p) defined on U such that the masses $\|T_n^\pm\|$ on U are uniformly bounded and that the $\mathscr{C}^3$-norms of $T_n^\pm$ are uniformly bounded near ∂B if $\partial B \neq \varnothing$. Note that if X is Kähler and T is of class $\mathscr{C}^3$ near ∂B, then the above mild condition is satisfied. In particular, this $\mathscr{C}^3$-smoothness near ∂B is automatically fulfilled if either $\partial B = \varnothing$ or $V \cap \mathrm{supp}(T) \subset B$.

- We define the notion of the j-th Lelong number of T along B for every j with $\max(0, l-p) \le j \le \min(l, k-p)$ and prove their existence as well as their basic properties. We also show that T admits tangent currents and that all tangent currents are not only positive plurisubharmonic, but also partially V-conic and partially pluriharmonic.
- When the currents $T_n^\pm$ are moreover pluriharmonic (resp. closed), we show, under a less restrictive smoothness of $T_n^\pm$ near ∂B, that every tangent current is also V-conic pluriharmonic (resp. V-conic closed).
- We also prove that the generalized Lelong numbers are intrinsic.
- In fact, if we are only interested in the top degree Lelong number of T along B (that is, the j-th Lelong number for the maximal value $j = \min(l, k-p)$), then under a suitable holomorphic context, the above condition on the uniform regularity of $T_n^\pm$ near ∂B can be removed.

Our method relies on some Lelong-Jensen formulas for the normal bundle to V in X, which are of independent interest.

The second part of our article is devoted to geometric characterizations of the generalized Lelong numbers. As a consequence of this study, we show that the top degree Lelong number of T along B is strongly intrinsic. This is a generalization of the fundamental result of Siu (for positive closed currents) and of Alessandrini–Bassanelli (for positive plurisubharmonic currents) on the independence of Lelong numbers at a single point on the choice of coordinates.

Keywords: positive plurisubharmonic currents, positive pluriharmonic currents, positive closed currents, tangent currents, Lelong-Jensen formula, the generalized Lelong numbers.

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