Operator theory in multiply connected domains, the similarity to a normal operator and resolvent growth

The talk is based on a joint work in progress with Glenier Bello (Univ. de Zaragoza).

Let $T : H \to H$ be a Hilbert space operator whose spectrum is contained in the closure Ω^{cl} of a finitely connected domain Ω with analytic boundary. A map $\Phi : \Omega^{\mathrm{cl}} \to \mathbb{D}^{\mathrm{cl}}$ will be called a Blaschke type function if it is holomorphic in Ω, maps Ω onto $\mathbb{D}$ and extends continuosly to a map from the boundary $\partial\Omega$ onto the unit circle $\partial\mathbb{D}$. Our motivation are two questions.

(1) Assuming that $\Phi(T)$ is similar to a contraction, prove that Ω^{cl} is a complete K-spectral set for T (in other words, T has a dilation, which is similar to a normal operator with spectrum in $\partial\Omega$).

(2) Assume that T has first order resolvent growth, that is, $\|(T-zI)^{-1}\| \le C \operatorname{dist}\big(z, \sigma(T)\big)^{-1}$. Prove (under some additional conditions) that T is similar to a normal operator.

The answer to Question 1 is positive. To prove it, we intoduce what we call an abstract input operator B and abstract output operator C for T. In case when $\Omega = \mathbb{D}$ and T is a contraction, one can take $B = D_{T^*}$ and $C = D_T$, where D_T, D_{T^*} are defect operators (but there is plenty of other choices). This gives rise to a kind of Nagy-Foias–like theory in a multiply connected domain.

To address Question 2, we study the relationship between the resolvent growth of T and that of $\Phi(T)$ (defined by the Riesz-Dunford calculus). We show that the answer to this question is positive if T has a finite rank input operator and $\sigma(T) \neq \Omega^{\mathrm{cl}}$. This is based on a result by Benamara and Nikolski concerning the case when $\Omega = \mathbb{D}$. To relate the resolvent growth of T with the resolvent growth of $\Phi(T)$, we use the techniques of Bakaev (1998) and the results by Y. Domar (1957) on the normality of families of subharmonic functions, defined by pointwice estimates on a domain in $\mathbb{C}$ and in $\mathbb{R}^k$. In fact, our results concern the resolvent growth of an analytic function f of a general linear operator T.