ECOLE DOCTORALE 631 MADIS

SUJET DE THÈSE EN MATHÉMATIQUE PROPOSÉ EN 2025

Title : Enumerative geometry of the gradient
Directeur de thèse : Mihai Tibăr — E-mail : mtibar@univ-lille.fr
Co-directeur de thèse : Cezar Joiţa, IMAR, Roumanie — E-mail : cezar.joita@imar.ro
Laboratoire : Paul Painlevé
Equipe : GT

The index of a vector field with isolated zeroes enters in the celebrated Poincaré-Hopf theorem which expresses the Euler characteristic of a compact manifold. It has been extended in various directions, in the real and in the complex geometry, by several authors such as Eisenbud-Levine, Khimshiashvili, etc.

In the complex setting, the index becomes a *degree*. For a holomorphic function germ h of $n \geq 2$ variables and with isolated singularity, the local index $\mathrm{ind}_p(\mathrm{grad}\ h)$ equals the *Milnor number* μ_h, which has several other topological and algebraic interpretations. In particular, μ_h is equal, modulo a sign, to the Euler characteristic of the Milnor fibre minus 1.

For any complex projective hypersurface $V \subset \mathbb{P}^n_{\mathbb{C}}$, defined by a homogeneous polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ of degree d, the *polar degree* $\mathrm{pol}(V)$ is defined as the topological degree of the gradient map, also known as the *Gauss map*:

$$\mathrm{grad} f : \mathbb{P}^n_{\mathbb{C}} \setminus \mathrm{Sing}\, V \to \mathbb{P}^n_{\mathbb{C}}. \tag{1}$$

The gradient mapping (1) with $\mathrm{pol}(V) = 1$ is a *Cremona transformation*. The corresponding hypersurfaces V were called *homaloidal*, and Dolgachev classified the projective plane curves with this property. The concept of polar degree goes back to 1851 when Hesse studied hypersurfaces with vanishing Hessian [Hes1, Hes2], which is equivalent to $\mathrm{pol}(V) = 0$, and to Gordan and Noether [GN].

The classification of all homaloidal hypersurfaces with *isolated singularities* was carried out by Huh[1] [Huh]. He confirmed a conjecture stated by Dimca and Papadima [DP] that there are no homaloidal hypersurfaces with isolated singularities besides the smooth quadric and the plane curves found by Dolgachev. June Huh proved in [Huh] the sum decomposition of the polar degree in the setting of V with isolated singularities:

$$\mathrm{pol}(V) = \mu_p^{\langle n-2\rangle}(V) + \mathrm{rank} H_n(\mathbb{P}^n \setminus V, (\mathbb{P}^n \setminus V) \cap H_p),$$

where $p \in \mathrm{Sing}\, V$ be one of the singular points of V, and $\mu_p^{\langle n-2\rangle}$ is the Milnor number of a hyperplane section $H_p \cap V$ through p.

[1] June Huh is a Fields medalist 2022.

More recently, Siersma-Steenbrink-Tibăr classified in [SST] the hypersurfaces with isolated singularities and polar degree 2, confirming Huh's conjectural list [Huh]. The finiteness of the range of (n, d) in which there may exist hypersurfaces with isolated singularities and polar degree $k > 2$ has been also proved in [SST].

This PhD project will address several open questions around the enumerative geometry of the degree (or the index) of the gradient, such as:

• The "admissible hyperplanes" defined in [ST] set the bases for an extended theory of geometric vanishing cycles. More other classical or recent problems may be approached with this new viewpoint.

• Study of the conjecture in the case $\dim \operatorname{Sing} V = 1$ that there are no other hypersurfaces with $\operatorname{pol} V = 0$ besides cones. (Conversely, all projective cones have $\operatorname{pol} V = 0$ by definition.)

• Classify the hypersurfaces $V \subset \mathbb{P}^n$ with $\dim \operatorname{Sing} V = 1$ which are homaloidal, i.e. such that $\operatorname{pol} V = 1$.

References

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