Ecole Graduée 631 MADIS

Sujet de thèse en Mathématique proposé en 2026

Titre : Low-lying zeros of L-functions associated to Bianchi modular forms

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Descriptif :

For a number field $K \setminus Q$, the discrete subgroup $\Gamma := \Gamma_K := PSL_2(O_K)$, where O_K denotes the ring of integers of K, is a co-finite subgroup of $PSL_2(C)$. The spectral theory of the hyperbolic Laplacian Δ on the hyperbolic space $M_\Gamma = \Gamma \setminus H^3$ is well-developed. In particular, as Γ is not co-compact but co-finite, the spectrum of $-\Delta$ consists of a discrete part of finite multiplicity and an absolutely continuous part. The discrete part features deeply arithmetic objects among which are the Bianchi modular forms, which are specific cases of automorphic forms $GL_2(A_K)$,themselves specific cases of automorphic representations of GL_2.

To such an automorphic form can be associated an L-function. The spacings of zeros of families of L-functions are well-understood: they are distributed according to a universal law, independent of the exact family under consideration, as proven by Rudnick and Sarnak. This recovers the behaviour of spacings between eigenangles of the classical groups of random matrices. However the distribution of *low-lying* zeros i.e. those located near the real axis, attached to reasonable families of L-functions does depend upon the specific setting under consideration. More precisely, let $L(s,f)$ be an L-function attached to an arithmetic object f. Consider its nontrivial zeros written in the $\rho_f = \frac{1}{2} + i\gamma_f$ where γ_f is a priori a complex number. There is a notion of analytic conductor c(f) of f quantifying the number of zeros of L(s,f) in a given region, and we renormalize the mean spacing of the zeros to 1 by setting $\widetilde{\gamma_f} = \log(c(f))\gamma_f / 2\pi$.

Let $h be an even Schwartz function on R whose Fourier transform is compactly supported, in particular it admits an analytic continuation to all C. The one-level density attached to f is defined by

$$D(f,h) := \sum_{\gamma_f} h(\widetilde{\gamma}_f).$$

The analogy with the behaviour of small eigenangles of random matrices led Katz and Sarnak to formulate the so-called desity conjecture, claiming the same universality for the types of symmetry of families of L-functions as those arising for classical groups of random matrices:

Conjecture (Katz-Sarnak) Let F be a family of L-functions and F_X a finite truncation increasing to F when X grows. Then there is one classical group G among U, SO(even), SO(odd), O or Sp such that for all even Schwartz function h on R with compactly supported Fourier transform,

$$\frac{1}{|F_X|}\sum_{f \in F_X} D(f,h) \longrightarrow \int_R h(x) W_G(x)\, dx,$$

where $W_G(x)$ is the explicit distribution function modellng the distribution of the eigenangles of the corresponding group of random matrices. The family F is then said to have the *type of symmetry* of G.

This conjecture is far from reach, but partial results do exist in some cases, with restrictions on the admissible test-functions h. This thesis would aim at proving such a result towards the Katz-Sarnak conjecture for Bianchi modular forms. This would build on the Weil explicit formulas that relate distribution on zeros to distribution of automorphic coefficients, and on trace formulas that relate such distribution of coefficients to arithmetic sums that can are better understood.

The limiting distribution displayed in the Katz-Sarnak conjecture display a transition phenomenon. Limiting statistics for densities of low-lying zeros around this critical point, beyond which symmetry is broken between different families, have been studied both numerically using elliptic curve databases and theoretically using explicit formulas. They feature specific oscillations uncovered in recent works, that can be interpreted as correlations between root numbers and automorphic coefficients. This phenomenon has been intensively active in the recent two years, and has been studied in very few cases: for elliptic curves by He et al., for modular forms by Zubrilina (in the level aspect) and for modular and Maass forms by Bober et al. (in the weight aspect). All these works rely on the Selberg trace formula, and are henceforth strongly tied to their specific settings. This thesis will as a second main direction explore an alternative proof using the relative trace formula, which is more robust with respect to the underlying group, and has been made very explicit in the case of the Bianchi modular group by Motohashi. This would be the first instance of such a phenomenon proven beyond $GL_2(R)$.