

ECOLE DOCTORALE 631 MADIS

SUJET DE THÈSE EN MATHÉMATIQUE PROPOSÉ EN 2024

Title : Enumerative geometry of the gradient

Directeur de thèse : Mihai Tibăr — E-mail : mtibar@univ-lille.fr

Co-directeur de thèse : Cezar Joița, IMAR, Roumanie — E-mail : cezar.joita@imar.ro

Laboratoire : Painlevé

Equipe : GT

A grant for 22 months may cover the first part of this PhD thesis, as part of the project “[Singularities and Application](#)” PNRR-III-C9-2023-I8 - Project CF 132/31.07.2023, founded by the European Union - NextGenerationEU - through Romania’s National Recovery and Resilience Plan, based at [IMAR](#).

The index of a vector field with isolated zeroes enters in the celebrated Poincaré-Hopf theorem which expresses the Euler characteristic of a compact manifold. It has been extended in various directions, in the real and in the complex geometry, by several authors such as Eisenbud-Levine, Khimshiashvili, etc.

In the complex setting, the index becomes a *degree*. For a holomorphic function germ h of $n \geq 2$ variables and with isolated singularity, the local index $\text{ind}_p(\text{grad } h)$ equals the *Milnor number* μ_h , which has several other topological and algebraic interpretations. In particular, μ_h is equal, modulo a sign, to the Euler characteristic of the Milnor fibre minus 1.

For any complex projective hypersurface $V \subset \mathbb{P}_{\mathbb{C}}^n$, defined by a homogeneous polynomial $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ of degree d , the *polar degree* $\text{pol}(V)$ is defined as the topological degree of the gradient map, also known as the *Gauss map*:

$$(1) \quad \text{grad } f : \mathbb{P}_{\mathbb{C}}^n \setminus \text{Sing } V \rightarrow \mathbb{P}_{\mathbb{C}}^n.$$

The gradient mapping (1) with $\text{pol}(V) = 1$ is a *Cremona transformation*. The corresponding hypersurfaces V were called *homaloidal*, and Dolgachev classified the projective plane curves with this property. The concept of polar degree goes back to 1851 when Hesse studied hypersurfaces with vanishing Hessian [[Hes1](#), [Hes2](#)], which is equivalent to $\text{pol}(V) = 0$, and to Gordan and Noether [[GN](#)].

The classification of all homaloidal hypersurfaces with *isolated singularities* was carried out by Huh¹ [[Huh](#)]. He confirmed a conjecture stated by Dimca and Papadima [[DP](#)] that

¹June Huh is a Fields medalist 2022.

there are no homaloidal hypersurfaces with isolated singularities besides the smooth quadric and the plane curves found by Dolgachev. June Huh proved in [Huh] the sum decomposition of the polar degree in the setting of V with isolated singularities:

$$\text{pol}(V) = \mu_p^{\langle n-2 \rangle}(V) + \text{rank} H_n(\mathbb{P}^n \setminus V, (\mathbb{P}^n \setminus V) \cap H_p),$$

where $p \in \text{Sing } V$ be one of the singular points of V , and $\mu_p^{\langle n-2 \rangle}$ is the Milnor number of a hyperplane section $H_p \cap V$ through p .

More recently, Siersma-Steenbrink-Tibăr classified in [SST] the hypersurfaces with isolated singularities and polar degree 2, confirming Huh’s conjectural list [Huh]. The finiteness of the range of (n, d) in which there may exist hypersurfaces with isolated singularities and polar degree $k > 2$ has been also proved in [SST].

This PhD project will address several open questions around the enumerative geometry of the degree (or the index) of the gradient, such as:

- The “admissible hyperplanes” defined in [ST] set the bases for an extended theory of geometric vanishing cycles. More other classical or recent problems may be approached with this new viewpoint.
- Study of the conjecture in the case $\dim \text{Sing } V = 1$ that there are no other hypersurfaces with $\text{pol } V = 0$ besides cones. (Conversely, all projective cones have $\text{pol } V = 0$ by definition.)

REFERENCES

- [DP] A. Dimca, S. Papadima, *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements*. Ann. of Math. (2) 158 (2003), no. 2, 473-507.
- [Do] I. Dolgachev, *Polar Cremona transformations*. Michigan Math. J. 48 (2000), 191-202.
- [GN] P. Gordan, M. Noether, *Über die algebraischen Formen, deren Hesse’sche Determinante identisch verschwindet*. Math. Annalen 10 (1876), 547-568.
- [Hes1] O. Hesse, *Über die Bedingung, unter welcher eine homogene ganze Function von n unabhängigen Variabeln durch lineäre Substitutionen von n andern unabhängigen Variabeln auf eine homogene Function sich zurück-führen läßt, die eine Variable weniger enthält*. Journal für die reine und angewandte Mathematik 42 (1851), 117-124.
- [Hes2] O. Hesse, *Zur Theorie der ganzen homogenen Functionen*. Journal für die reine und angewandte Mathematik 56 (1859), 263-269.
- [Huh] J. Huh, *Milnor numbers of projective hypersurfaces with isolated singularities*. Duke Math. J. 163 (2014), no. 8, 1525-1548.
- [SST] D. Siersma, J.H.M. Steenbrink, M. Tibăr, *On Huh’s conjectures for the polar degree*, J. Algebraic Geometry 30 (2021), 189-203.
- [ST] D. Siersma, M. Tibăr, *Polar degree and vanishing cycles*, J. Topology 15 (2022), no. 4, 1807-1832.
- [Ti3] M. Tibăr, *Polynomials and vanishing cycles*. Cambridge Tracts in Mathematics, 170. Cambridge University Press, Cambridge, 2007.